High-Rate Vector Quantization for the Neyman-Pearson Detection of Some Stationary Mixing Processes

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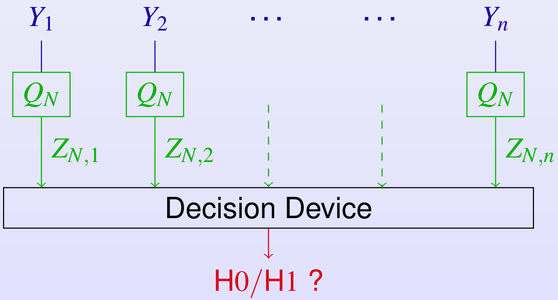
Introduction

- $Y_{1:n} = (Y_1 \dots Y_n)$: a stationary vector-valued process
- Binary test
  $$\text{H0} : Y_{1:n} \sim p_0$$
  $$\text{H1} : Y_{1:n} \sim p_1$$



Introduction (cont.)

Our aims:

- evaluate the performance of the test
- determine relevant quantization rules

Main difficulties:

- quantization is a complex operation
- observations are correlated

Outline

1. Detection from Unquantized Obervations
2. Detection from Quantized Obervations
3. Detection in the High-Rate Regime

Neyman-Pearson Hypothesis Testing

- $Y_{1:n} = (Y_1 \dots Y_n)$: a stationary vector-valued (in $\mathbb{R}^d$) Lebesgue-dominated process

- Binary test
  $$\begin{aligned} \text{H0} &: Y_{1:n} \sim \mathbb{P}_0 \ (\text{pdf } p_0) \\ \text{H1} &: Y_{1:n} \sim \mathbb{P}_1 \ (\text{pdf } p_1) \end{aligned}$$

Neyman-Pearson Hypothesis Testing

- $Y_{1:n} = (Y_1 \dots Y_n)$: a stationary vector-valued (in $\mathbb{R}^d$) Lebesgue-dominated process

- Binary test $\quad$ H0 : $Y_{1:n} \sim \mathbb{P}_0$ (pdf p_0)
  H1 : $Y_{1:n} \sim \mathbb{P}_1$ (pdf p_1)

Neyman-Pearson strategy:

- set $\mathbb{P}_0$(decide H1) $= \alpha$ *false alarm*
- minimize $\mathbb{P}_1$(decide H0) $\rightarrow \beta_n(\alpha)$ *miss*

Likelihood Ratio Test: $$L_n = \log \frac{p_1}{p_0}(Y_{1:n}) \underset{\text{H0}}{\overset{\text{H1}}{\gtrless}} \gamma$$

Error Exponent

Our aim is to measure the detection performance.

- $\beta_n(\alpha)$ is a good performance measure ...
- ... but is not tractable

$\rightarrow$ asymptotic regime $n \rightarrow \infty$

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Lemma (Stein-Chen)

If $\exists\, K > 0$ *such that* $(-1/n)L_n \xrightarrow{P} K$ *under* H0 *then*

$$\forall\, \alpha \in (0,1) \qquad \lim_{n \to +\infty} -\frac{1}{n} \log \beta_n(\alpha) = K$$

K is the error exponent of the test: $\beta_n(\alpha) \approx \exp(-nK)$

Error Exponent with Perfect Observations ($n \to \infty$)

Assumption

$(\log p_i(Y_0|Y_{-m:-1}))_{m\geq 0}$ is a convergent sequence in $L^1(\mathbb{P}_0)$.

e.g. valid for a wide class of hidden Markov models.

Shannon-McMillan-Breiman–like result

The normalized LLR $-(1/n)L_n$ converges under H0 to

$$K = \mathbb{E}_0\left[\log \frac{p_0}{p_1}(Y_0|Y_{-\infty:-1})\right]$$

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Neyman-Pearson Test on Quantized Obervations

- Quantized observation: $Z_{N,k} = Q_N(Y_k)$

- The test becomes: $\text{H0} : Z_{N,1:n} \sim p_{0,N}$
  $\text{H1} : Z_{N,1:n} \sim p_{1,N}$

Error exponent

$$K_N = \mathbb{E}_0 \left[\log \frac{p_{0,N}}{p_{1,N}} (Z_{N,0} | Z_{N,-\infty:-1}) \right]$$

Our aim is to study the error exponent loss $K - K_N$

Effect of the Quantization Rule

- a quantizer = a partition of $\mathbb{R}^d$
- the error exponent loss is not directly informative

$$K - K_N = \mathbb{E}_0\left[\log\frac{p_0}{p_1}(Y_0|Y_{-\infty:-1})\right] - \mathbb{E}_0\left[\log\frac{p_{0,N}}{p_{1,N}}(Z_{N,0}|Z_{N,-\infty:-1})\right]$$

→ special cases:

- $N = 2$
- $N \to \infty$ (high-rate quantization)

[Gupta & Hero – 2003] for i.i.d. observations.

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High-Rate Quantization ($N \to \infty$)

cf. [Bennett48], [Gray98]

model point density $\zeta(y)$

$\approx$ asymptotic number of cells in the neighborhood of y

In the high-rate regime:

$$\frac{\text{number of quantization points in } A}{N} \longrightarrow \int_A \zeta(y)\, dy$$

High-Rate Quantization ($N \to \infty$)

cf. [Bennett48], [Gray98]

model point density $\zeta(y)$

$\approx$ asymptotic number of cells in the neighborhood of y

model covariation profile $M(y)$

$$= \lim_{N\to\infty} \frac{1}{V_N(y)^{1+2/d}} \int_{C_N(y)} (s - Q_N(y))(s - Q_N(y))^{\mathsf{T}} ds \,.$$

- a matrix-valued function. . .
- . . . which provides information about the shape of the cells

Functions ζ and M completely characterize the quantizer.

Main Result

Theorem (Asymptotic Error Exponent Loss)

$$N^{2/d}(K - K_N) \xrightarrow[N\to\infty]{} D = \frac{1}{2}\int \frac{p_0(y)F(y)}{\zeta(y)^{2/d}}\,dy$$

where

$$F(y) = \mathbb{E}_0\left[\nabla_{y_0}\log\frac{p_0}{p_1}(Y_{-\infty:\infty})^{\mathsf{T}} M(Y_0)\,\nabla_{y_0}\log\frac{p_0}{p_1}(Y_{-\infty:\infty})\,\middle|\, Y_0 = y\right]$$

Under the mixing condition:

$$\mathbb{E}_0\left|\log p_i(Y_0|Y_{-m:-1}) - \log p_i(Y_0|Y_{-m-\ell:-1})\right| \;=\; O(m^{-6-\epsilon})\;,$$

and some good smoothing conditions on the log-densities.

Key Ideas of the Proof

- $$(K - K_N) = \lim_{m\to\infty} \mathbb{E}_0 \left[\log \frac{p_0}{p_1}(Y_0|Y_{-m:-1}) - \log \frac{p_{0,N}}{p_{1,N}}(Z_{N,0}|Z_{N,-m:-1})\right]$$

Key Ideas of the Proof

- $$\lim_{N\to\infty} N^{2/d}(K - K_N)$$
  $$= \lim_{N\to\infty} \lim_{m\to\infty} N^{\frac{2}{d}} \mathbb{E}_0 \left[\log \frac{p_0}{p_1}(Y_0|Y_{-m:-1}) - \log \frac{p_{0,N}}{p_{1,N}}(Z_{N,0}|Z_{N,-m:-1})\right]$$

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- Taylor-Lagrange expansion of densities: $\frac{p_{0,N}}{p_{1,N}} \approx \frac{p_0}{p_1}$ as $N \to \infty$

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Main issue:

Find relevant estimates of the remainders in m, N.
$\to$ Mixing conditions are needed.

Determination of Relevant Quantization Rules

$\rightarrow$ Find (ζ, M) which minimizes the loss D:

$$D = \frac{1}{2} \int \frac{p_0(y) F(y)}{\zeta(y)^{2/d}} \, dy$$

$$F(y) = \mathbb{E}_0 \left[\nabla_{y_0} \log \frac{p_0}{p_1}(Y_{-\infty:\infty})^{\mathsf{T}} M(Y_0) \, \nabla_{y_0} \log \frac{p_0}{p_1}(Y_{-\infty:\infty}) \,\middle|\, Y_0 = y \right]$$

Determination of Relevant Quantization Rules

$\rightarrow$ Find (ζ, M) which minimizes the loss D:

- Scalar case ($d = 1$): optimal regular quantizer, $M(y) = \frac{1}{12}$

$$\zeta^*(y) = \frac{\left[p_0(y)\bar{F}(y)\right]^{1/3}}{\int \left[p_0(s)\bar{F}(s)\right]^{1/3} ds}$$

$$\bar{F}(y) = \mathbb{E}_0\left[\left(\frac{\partial}{\partial y_0}\log\frac{p_0}{p_1}(Y_{-\infty:\infty})\right)^2 \middle| Y_0 = y\right]$$

Determination of Relevant Quantization Rules

$\rightarrow$ Find (ζ, M) which minimizes the loss D:

- Scalar case ($d = 1$): optimal regular quantizer, $M(y) = \frac{1}{12}$
- Vector case ($d \geq 2$):

  classical algorithms (*e.g*, Linde-Buzo-Gray): $M(y) = \upsilon I_d$

  $\rightarrow$ "locally" optimal quantizer

$$\zeta^*(y) = \frac{\left[p_0(y)\bar{F}(y)\right]^{d/(d+2)}}{\int \left[p_0(s)\bar{F}(s)\right]^{d/(d+2)} ds}$$

$$\bar{F}(y) = \mathbb{E}_0\left[\left\|\nabla_{y_0} \log \frac{p_0}{p_1}(Y_{-\infty:\infty})\right\|^2 \,\middle|\, Y_0 = y\right]$$

Detection of a 2-D Gaussian AR-1 Structure

State:

H0 : $X_k \overset{i.i.d.}{\sim} \mathcal{CN}(0,1)$

H1 : $X_k = aX_{k-1} + \sqrt{1-a^2}\, U_k$

$a \in (0,1)$: correlation coefficient

$U_k \overset{i.i.d.}{\sim} \mathcal{CN}(0,1)$: innovation

Observation:

$$Y_k = X_k + W_k$$

$W_k \overset{i.i.d.}{\sim} \mathcal{CN}(0,\sigma^2)$: obs. noise

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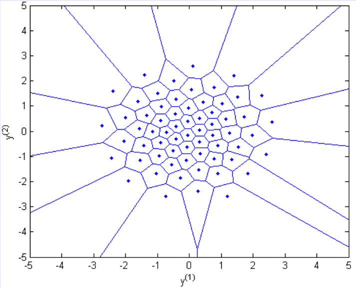
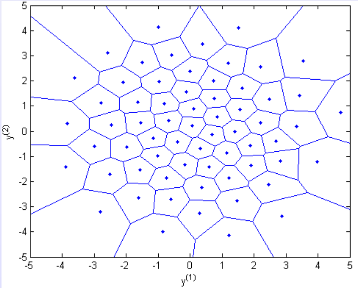


MSE-optimal 64-cell quantizer ($a = 0.8$, $\sigma = 1$)



Proposed 64-cell quantizer ($a = 0.8$, $\sigma = 1$)

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Error exponent loss ($a = 0.8$, $\sigma = 1$):

Quantization rule	Uniform on $[-8;8]^2$	MSE-optimal	Proposed one
Loss D	8.211	2.255	2.112

Conclusion

- Neyman-Pearson test on quantized observations
  $\rightarrow n$ observations, quantization on $\log_2(N)$ bits

- Evaluation of the performance

$$\beta_n(\alpha) \approx e^{-n\left(K - \frac{D}{N^{2/d}}\right)}$$

  for large n, N and $n \gg N$.

- Optimal scalar,
  "locally" optimal vector quantization rules

- Valid for a class of stationary mixing processes

Conclusion (cont.)

Extended version (with complete proofs and more examples):

- submitted to *IEEE Trans. Inf. Theory*,
- available on *Arxiv* (arXiv:1004.5529).

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Thank you for your attention.