High-Rate Quantization for the Neyman-Pearson Detection of Hidden Markov Processes

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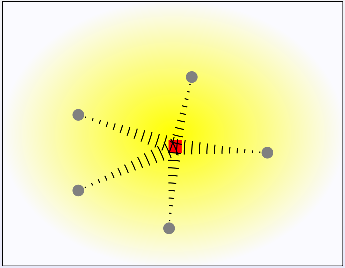
ITW 2010

Context

- a physical phenomenon
  with space correlation
- some sensors
- a fusion center
- wireless channels
  $\rightarrow$ quantization



Goal: detection from quantized observations

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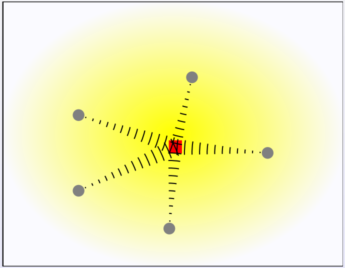

Goal: detection from quantized observations

Questions:

- performance of the Neyman-Pearson test?
- best quantization?

- Neyman-Pearson test on quantized observations
  - n sensors
  - quantization on $\log_2(N)$ bits

- Evaluation of the performance:

$$P_e \approx e^{-n\left(K - \frac{D}{N^2}\right)}$$

for large n, N and $n \gg N$.

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Our aims:

- Evaluate the loss D due to quantization
- Find the best quantization rule

Outline

1. Detection from Unquantized Obervations
2. Detection from Quantized Obervations
3. Detection in the High-Rate Regime

Neyman-Pearson Hypothesis Testing

- $Y_{1:n} = (Y_1 \ldots Y_n)$: a stationary real-valued Lebesgue-dominated process with mixing properties

- Binary test $$\begin{aligned} H_0 &: Y_{1:n} \sim p_0 \\ H_1 &: Y_{1:n} \sim p_1 \end{aligned}$$

Neyman-Pearson Hypothesis Testing

- $Y_{1:n} = (Y_1 \dots Y_n)$: a stationary real-valued Lebesgue-dominated process with mixing properties
- Binary test $\quad H_0 : Y_{1:n} \sim p_0$
  $\quad\quad\quad\quad\quad H_1 : Y_{1:n} \sim p_1$

Neyman-Pearson strategy :

- set $P_{H_0}(\text{decide } H_1) = \alpha$ *false alarm*
- minimize $P_{H_1}(\text{decide } H_0) \rightarrow \beta_n(\alpha)$ *miss*

Likelihood Ratio Test :
$$L_n = \frac{1}{n} \log \frac{p_0}{p_1}(Y_{1:n}) \underset{H_1}{\overset{H_0}{\gtrless}} \lambda_n$$

Error Exponent

Our aim is to measure the detection performance.

- $\beta_n(\alpha)$ is a good performance measure ...
- ... but is not tractable

$\rightarrow$ asymptotic regime $n \rightarrow \infty$

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Lemma (Stein – Chen)

If $\exists\, K > 0$ *such that* $L_n \xrightarrow{P} K$ *under* H_0 *then*

$$\forall\, \alpha \in (0,1) \qquad \lim_{n\to+\infty} \frac{1}{n} \log \beta_n(\alpha) = -K$$

K is the error exponent of the test: $\beta_n(\alpha) \approx \exp(-nK)$

Error Exponent for Unquantized Observations ($n \to \infty$)

Under a certain mixing condition on p_1

(*e.g.* valid for a wide class of hidden Markov models)

Shannon-McMillan-Breiman–like result

The LLR L_n converges under H_0 to

$$K = E_0 \left[\log \frac{p_0}{p_1} (Y_0 | Y_{-\infty:-1}) \right]$$

K is the error exponent of the NP test.

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Neyman-Pearson Test on Quantized Obervations

- Quantized observation: $Z_{N,k} = Q_N(Y_k)$
- The test becomes:

$$H_0 : Z_{N,1:n} \sim p_{0,N}$$
$$H_1 : Z_{N,1:n} \sim p_{1,N}$$

Error exponent

$$K_N = E_0 \left[\log \frac{p_{0,N}}{p_{1,N}} (Z_{N,0} | Z_{N,-\infty:-1}) \right]$$

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Our aim is to study the error exponent loss $K - K_N$

$\rightarrow$ [Gupta & Hero – 2003] for i.i.d. observations.

Outline

High-Rate Quantization ($N \to \infty$)

Asymptotic regime: $n, N \to \infty$ but $n \gg N$

cf. [Bennett48], [Gray98]

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model point density ζ

$\approx$ asymptotic number of cells in the neighborhood of y

In the high-rate regime:

$$\frac{\text{number of quantization points in } A}{N} \Rightarrow \int_A \zeta(y)\, dy$$

Main Result

Theorem (Asymptotic Error Exponent Loss)

$$N^2(K - K_N) \xrightarrow[N\to\infty]{} D_\zeta = \frac{1}{24}\int \frac{p_0(y)F(y)}{\zeta(y)^2}\,dy$$

where $F(y) = E_0\left[\left(\frac{\partial}{\partial y_0}\log\frac{p_0}{p_1}(Y_{-\infty:\infty})\right)^2 \,\middle|\, Y_0 = y\right]$

Under some mixing conditions:

$$\eta_m^{-1} \le \frac{p_i(Y_0|Y_{-m':-1})}{p_i(Y_0|Y_{-m:-1})} \le \eta_m\,, \quad \eta_m^{-1} \le \frac{p_{i,N}(Z_{N,0}|Z_{N,-m':-1})}{p_{i,N}(Z_{N,0}|Z_{N,-m:-1})} \le \eta_m$$

$$\left|\frac{\partial}{\partial y_0}\log p_i(Y_{0:k}|Y_{-\ell:-1}) - \frac{\partial}{\partial y_0}\log p_i(Y_{0:k}|Y_{-\ell':-1})\right| \le \varphi_\ell$$

$$\left|\frac{\partial}{\partial y_0}\log p_i(Y_k|Y_{-\ell:k-1})\right| \le \psi_k$$

for $\log\eta_m = O(m^{-6-\varepsilon})$ and some summable φ_k and ψ_k.

Key Ideas of the Proof

- Inversion of two limits:

$$\lim_{N\to\infty} N^2(K - K_N)$$
$$= \lim_{N\to\infty} \lim_{m\to\infty} N^2 E_0\left[\log \frac{p_0}{p_1}(Y_0|Y_{-m:-1}) - \log \frac{p_{0,N}}{p_{1,N}}(Z_{N,0}|Z_{N,-m:-1})\right]$$
$$\stackrel{?}{=} \lim_{m\to\infty} \lim_{N\to\infty} N^2 E_0\left[\log \frac{p_0}{p_1}(Y_0|Y_{-m:-1}) - \log \frac{p_{0,N}}{p_{1,N}}(Z_{N,0}|Z_{N,-m:-1})\right]$$

- Taylor-Lagrange expansion of densities: $\frac{p_{0,N}}{p_{1,N}} \approx \frac{p_0}{p_1}$ as $N \to \infty$

Main issue:

Find relevant estimates of the remainders in m, N.
$\to$ Mixing conditions are needed.

Detection of a Gaussian AR-1 Process in Noise

State :

$$X_k = aX_{k-1} + \sqrt{1-a^2}\, U_k$$

$a \in (0,1)$: correlation coefficient
$U_k \overset{i.i.d.}{\sim} \mathcal{N}(0,1)$: innovation

Observation :

$$H_0 : \; Y_k = W_k$$
$$H_1 : \; Y_k = X_k + W_k$$

$W_k \overset{i.i.d.}{\sim} \mathcal{N}(0,\sigma^2)$: obs. noise

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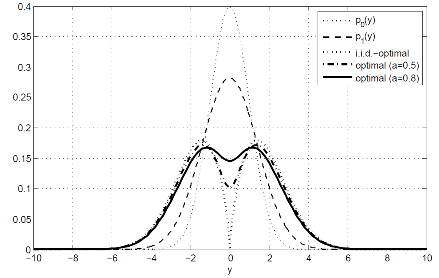

Probability and model point densities ($\sigma = 1$)

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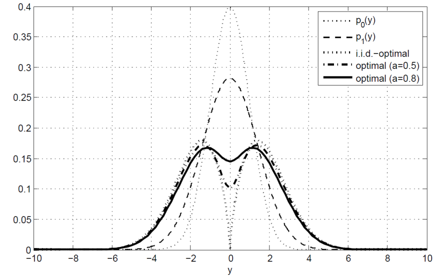

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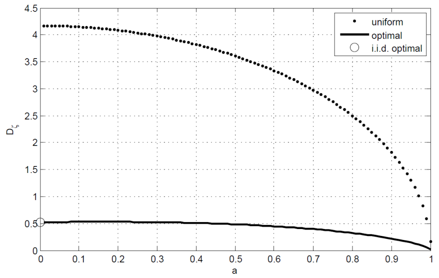

$D_\zeta = f(a)$ for different quantization strategies ($\sigma = 1$)

Conclusion

- Neyman-Pearson test on quantized observations
  $\rightarrow n$ sensors, quantization on $\log_2(N)$ bits

- Evaluation of the performance:

$$\beta_n(\alpha) \approx e^{-n\left(K - \frac{D}{N^2}\right)}$$

for large n, N and $n \gg N$.

- Optimal quantization rule

$\rightarrow$ Valid for a wide class of stationary mixing processes

Ongoing Work: Vector Quantization (to be sub. to ISIT2010)

Vector-valued process: $Y_k \in \mathsf{Y} \subset \mathbb{R}^d$

Theorem (Asymptotic Error Exponent Loss)

Under some mixing conditions:

$$N^{2/d}(K - K_N) \xrightarrow[N\to\infty]{} D_e = \frac{1}{2}\int \frac{p_0(y)F(y)}{\zeta(y)^{2/d}}\,dy\ ,$$

where $F(y) = \mathbb{E}_0\left[\nabla_{y_0}\log\frac{p_0}{p_1}(Y_{-\infty:\infty})^{\mathsf{T}} M(Y_0)\,\nabla_{y_0}\log\frac{p_0}{p_1}(Y_{-\infty:\infty})\,\middle|\,Y_0 = y\right]$

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M is the model covariation profile:

- a matrix-valued function…
- …which provides information about the shape of the cells

Thank you for your attention.